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# ON THE UNITY OF OPPOSITES, AS A WAY TO ATTAIN HYPOTHESES

Memoria inviata dal *Socio straniero dell'Accademia* Enric Trillas, Oviedo (Asturias), Spain

RIASSUNTO

In questo lavoro viene presentato uno studio del metodo dialettico effettuato mediante un modello formale del ragionamento di senso comune strettamente connesso al linguaggio ordinario. È un modello nel quale, a partire da poche definizioni, viene soddisfatto un numero ristretto di leggi alle quali se ne possono aggiungere altre quando queste sono effettivamente presenti nel linguaggio. Il modello è quindi uno "scheletro formale" del ragionamento mediante il quale, però, nonostante la sua semplicità, si riescono già a dimostrare alcuni risultati significativi; per esempio, quando le conseguenze e le ipotesi sono congetture, che il dedurre e il confutare sono monotone, mentre l'abdurre e il congetturare sono antimonotone così come i principi aristotelici di non contraddizione e del terzo escluso.

Questo scheletro formale mostra che la congiunzione linguistica di un enunciato (tesi) e di uno dei suoi opposti (antitesi) produce una ipotesi (sintesi). Una spiegazione della tesi che, sotto condizioni molto deboli, risulta essere una congettura: la "congiunzione degli opposti" permette di congetturare una prima ipotesi della tesi. Si analizza anche dove la congiunzione degli opposti è effettiva, come determinare un insieme sia esso classico o sfocato (fuzzy) nel quale realmente, avvenga; senza di esso la 'congiunzione' non ha significato.

Da quanto detto si coglie emerge che la *coincidenza degli opposti* **non** è un 'principio' ma un metodo per congetturare una prima ipotesi: La metodologia usata nel lavoro rispetta il principio di Occam-Menger; *"Non usare più di quanto è necessario né meno di quanto è indispensabile per potere ottenere risultati significativi"*.

## ON THE *UNITY OF OPPOSITES*, AS A WAY TO ATTAIN HYPOTHESES

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*<This paper is dedicated to Professor Settimo Termini>*

## 1. INTRODUCTION

What follows tries to analyze the old, medieval, *Coincidentia Oppositorum,* the coincidence of opposites, usually translated into English as the *Unity of Opposites* [1],and from the point of view of ordinary, or commonsense reasoning [2], but not from a Pure Logic's perspective [13,14,15]. Coincidence of opposites is actually essential for what concerns Dialectic Materialism in which, for instance, Vladimir Ilyich Ulyanov, *Lenin,* stated it is the most important Dialectical 'Principle' Marxian Dialectics is based on [1].

Foremost, it should be pointed out that the Unity of Opposites cannot refer to the 'identity' between a statement and one of its opposites, or antonyms. In fact, were $p = p^a$ with p a statement and $p^a$ one of its opposites, from the inferential 'relation of coherence' [2,9], $p^a < p'$, 'If antonym of p, then negation of p', that is, p refutes $p^a$, it will follow, under transitivity in the triplet $(p, p^a, p')$, the absurd $p < p'$, that p is self-contradictory, self-refuting, or 'impossible' [2]. It will mean that the departure's knowledge given by statement p is necessarily absurd, is a non sense. Notice that without local transitivity in that triplet, the former proof can't be obtained, and that, analogously, $p < p^a$ conducts to the same absurd.

Out of Paraconsistent Logics [18] whose interest lies, possibly, in jointly managing contradictory hypotheses under a complete deductive axiomatic regulation, and without taking into account conjecturing, as well as also out of the so-called Logics of the Contradictory [13,14] mainly considered by Marxian-oriented thinkers, self-contradictory premises or conclusions are always avoided.

*Coincidentia oppositorum*, actually introduced by the Greek philosopher Heraclitus of Ephesus (535-475 BC), was methodologically managed, in his theological writings, by Cardinal Nicholas of Cusa in the 13th Century and, mainly, in his books *De Docta Ignorantia*, and *De coniecturis.* Further, the methodology was used in the 18th Century by Friedrich Hegel in his philosophy, and later on was adopted by Karl Marx and Friedrich Engels in the 19th Century to develop Dialectical Materialism [13,14].

For what has been said, such *coincidentia* does refer to something different to the identity $p = p^a$, like it can be conjunction, or some other kind of coincidence. This paper tries to analyze, in the first place and in an ordinary or commonsense reasoning's formal framework, the linguistic conjunction 'p and $p^a$', shortened by $p \cdot p^a$. For it, expressing $p^a$ through p, that is, reducing the analysis to what expresses the premise p, seems interesting; it will be done thanks to what was advanced in references [5] and [9].

In any case, such view on the Unity of Opposites requires to be based in a 'common ground' between p and $p^a$ in the universe on which p states something. A ground that, further considered in this paper, can allow to 'keep the feet in the floor', on a solid basement where the attained conclusions actually can hold. Concerning what *coincidentia* can allow to conclude, what can be said on the conjunction $p \cdot p^a$?

The former comments, and also what will follow, concern a naïve formal view of *coincidentia oppositorum* in the open setting of ordinary or commonsense reasoning, but not in a closed one where logical deduction is done step-by-step by just applying a list of operative axioms [13]. Basically, what this paper tries to show is that *the Unity of Opposites is not a 'principle' of reasoning, but a method for attaining hypotheses, explanations*; a method that, in addition and as it will be shown, is not unique but shows that Cusa, Hegel, Marx, Engels, etc., were not wrong using it as a way to explain something.

From a different perspective and motivation, the Unity of Opposites can be seen within a human historical view, like that sustained by the Marxian Dialectical Materialism, depending on 'time' and the 'opposed forces' intervening at each situation and moment, and where the coincidence of opposites refers to some internal and opposite tensions. In such perspective, the corresponding analysis can't be conducted [13,14] by just the formal 'reasoning's skeleton' here presented that, previously introduced by the author in references [2], [8] and [16], and even covering many linguistic cases can serve, at most, as a theoretic basement helping, perhaps, to clarify such view but without necessarily presuming laws or axioms not always manifest in ordinary language and commonsense reasoning.

Working in the framework facilitated by such formal skeleton comes from the author's agreement with the Ockham-Menger's Razor: To explain something neither presume more laws than necessary (William of Ockham, 14th Century), nor less than those sufficient for obtaining significant conclusions (Karl Menger, 20th Century) [8,16]. The Razor's methodological rule was expressed by Albert Einstein with the nice words, 'Everything should be made as simple as possible, but not simpler'.

## 2. A FORMAL SKELETON OF REASONING

2.1. Let's shortly introduce the necessary formal concepts on ordinary or commonsense reasoning [2,8,16], constituting a mathematical mode the author likes to call the *Formal Skeleton* of reasoning.

Reasoning is a specialization of thinking with the goal of attaining *conclusions* (statements q, r, s, etc.) from a departing information summarized by a not self-contradictory statement, or *premise*, p. Reasoning is based in the binary, linguistic and primitive, relation < of *inference* that, shortening the linguistic conditional statement ‘If p, then q’ by p < q, is only (universally) submitted to be reflexive, that is, r < r for all statement r. Usually, < is not symmetric, when it is p < q it is not usually q < p, and reciprocally; when both hold it is said that p and q are inferentially *equivalent*, and it is written p ~ q. When it is neither p < q, nor q < p, it is said that such statements are *orthogonal*, and it is written p ◊ q.

Obviously, relation ~ is reflexive and symmetric, but ◊ is never reflexive and only is symmetric. Provided < were transitive, that is, [p < q and q < r imply p < r], also ~ will be transitive, but nothing at the respect can be stated on ◊. Notice that the transitive property of < is often *local* in each triplet (p, q, r); it is only universal whenever it holds for all triplet of statements, a property that can’t be always presumed in ordinary or commonsense reasoning, usually expressed in a natural language, and as trivial examples show [16]. Relation <, appearing in language, is here considered as primitive, given, and like in the Euclid’s *Elements* are taken as primitive ‘point’, ‘line’, etc.

It should be pointed out that there is a reason for considering < undefined, primitive. In language, conditional statements If/then are not always understood in the same form [19]; even < is local. If, for instance, in classical logic ‘If p, then q’ is considered affirming the unconditional statement ‘not p or q’, and in Quantum Logic is often seen as affirming ‘not p or (p and q)’, in ordinary language is also, and sometimes understood as ‘p and q’. This last linguistic interpretation forces to remember that in language conjunction is not always commutative.

Notice that the reflexive law of < states that it *actually exists*, that it is not empty, since at least contains the pairs (r, r) for all statement r. To be sure that < is *effective* for inference, is supposed it verifies the rule *Modus Ponens*: Once p and p < q are known, also q is known, that is, relation < conducts effectively from p to q. The name Modus Ponens comes from the Latin *Modus Ponendo Ponens,* the mode of posing q after posing p. Both the universal reflexive law, and this (also universal) rule, constitute a minimal guarantee for inference; avoiding reflexivity and effectiveness can imply that sometimes, and in posing p < q, the consequent q actually will remain unknown. Both are implicitly supposed to hold in the 'skeleton'.

2.3. It is supposed that for each r with a *negation* $\text{not-}t = r'$, this negation is unique and only submitted to verify the (universal) law: $p < q \Rightarrow q' < p'$, provided both p and q have a negation. Negation always reverses the inference relation, and, when existing, belongs to one of the following types: 1) *Weak* at r, when $r < (r')'$; 2) *Intuitionistic* at r, when $(r')' < r$; 3) *Strong* at r, when is both weak and intuitionistic at r, that is, when $r \sim (r')'$; 4) *Wild* at q, when it is $r \lozenge (r')'$. In ordinary language and commonsense reasoning, the type of negation is also not universal, is just local. It is said that r *refutes* p, is a *refutation* of p, if $p < r'$. In the old Greek Philosophy, refutation is the first step in the Socratic argumentation.

From now on, when writing r' it will be supposed that r is with negation as it is usual in natural language. A typical example in which negation is not universally existing is given by the programming language *Prolog*, in which and notwithstanding, when negations r' and (r')' do exist, verify $(r')' < r$, that is, the negation (by failure) is intuitionistic in r.

Independently of the negation's type, from the rule Modus Ponens it follows the rule *Modus Tollens*: q' and $p < q$, imply p' if it exists, In fact, since $p < q \Rightarrow q' < p'$, given q' and $p < q$, it follows q' and $q' < p'$, and thus p'. The name Modus Tollens comes from the Latin *Modus Tollendo Tollens,* the mode of removing p after removing q.

2.4. When some q is found verifying $p < q$, q is a *consequence* of p; under transitivity, no two consequences with negation can be contradictory: Were it $p < q$, $p < q^*$, and $q^* < q'$, as $p < q \Rightarrow q' < p'$, provided < is transitive it follows the absurd $p < p'$. In particular, and provided $p \not< p'$, no consequence of p is self-contradictory.

When h verifies $h < p$, h is a *hypothesis* for p; contrary to the case of consequences, the existence of contradictory hypotheses is but an empirical evidence. Only not self-contradictory hypotheses are considered; in reasoning, self-contradiction is seen as a kind of 'Death Sin', a border that never should be trespassed: Provided someone is ‘buying flowers and not buying flowers’, what is she/he doing, may be is ‘looking for flowers’ or just walking and looking around? Self-contradictory statements seem to paralyze the actions usually derived from commonsense reasoning.

2.5. It is said that q is a *conjecture* from p, when $p </q'$, that is, when q does not refute p. Most of the usual conclusions reached by people are but conjectures, and with transitivity, both consequences and non self-contradictory hypotheses, are conjectures:

1) If $p < q$, and if it were $p < q'$, since from the first it follows $q' < p'$, it will result the absurd $p < p'$. Thus, it is $p </q'$.

2) If $h < p$, and $p < h'$, then follows the absurd $h < h'$; thus, it should be $p </h'$.

There exists a third type of conjectures, those orthogonal to p, that is, those statements s such that $p </ s'$ and $p \lozenge s$. Such conjectures are called *speculations* from p, and they belong to the following two excluding types:

a) $p \lozenge e$, and $e' < p$, called *weak* speculations, and,

b) $p \lozenge s$ and $p \lozenge s'$, called *strong* speculations.

Not generally recognized in the logical literature, speculations were formally introduced in reference [3] and in the setting of Ortho-complemented lattices.

By its side, weak speculations (with negation and double negation) can be inferentially attained unless the negation is wild at them: In fact, $s' < p$ shows that s’ is backwards attainable from p, and once s’ is known, it suffices to negate it, obtaining (s’)’, and then: If the negation is weak, backwards is attained s, $s < (s')'$, and if the negation is intuitionistic, s is forwards attained, $(s')' < s$. Notice that s is finally reached through either forwards or backwards inference.

At its turn, the two conditions defining strong speculations, $p \lozenge s$ and $p \lozenge s'$, don’t allow to repeat the former argumentation; strong speculations are neither attainable, separately, by *deduction* (forwards inference), nor by *abduction* (backwards inference); the orthogonal character of p with both s and s’, shows it.

Weak speculations under a wild negation and strong speculations, are *guesses,* the *inductive* part of reasoning, or *guessing*; finding them is done by *induction.* Nevertheless, it not excludes the possibility of attaining them by mixing either forwards/backwards, or backwards/forwards chains of inference, that is, by a non separate mixture, or zigzag, of deduction and abduction; something that, up to now only proven in the case of finite Boolean algebras [4], seems to announce possible ways toward 'mechanizing' guessing.

2.6. A linguistic term q is called *regular* if it has some opposite linguistic term $q^a$, or antonym; if negation is unique, *opposites* or *antonyms* are not always so, often there are more than one [9]. Opposites are supposed to verify the *coherence* relation [9,2] $p^a < p'$, showing that p refutes $p^a$; what is not always the case is $p' < p^a$, that is $p^a \sim/ \ p'$. Nevertheless, it is usually presumed $(p^a)^a \sim p$, that opposition is involutive.

For what concerns conjunction (symbolized by a dot (·)), and disjunction (symbolized by a cross (+)), they are only supposed to universally verify the laws: $p \cdot q < p$, $p \cdot q < q$, and $p < p + q$, $q < p + q$. Notice that no other laws, like the commutative or the associative, are presumed; additional laws, with a local character, are only presumed when they manifestly appear in language and reasoning. Among such laws, those of duality: $(p \cdot q)' = p' + q'$, and $(p + q)' = p' \cdot q'$, can be cited but remembering that in commonsense reasoning are not necessarily equivalent, that (·) and (+) are independent, that no one, or one but not the other, can hold [16]. In commonsense reasoning such laws are just of a local character.

## 3. WHEN 'COINCIDENCE' IS POSSIBLE, AND WHAT IS IT IN REASONING

3.1. In the case p is not regular, that is, it has no any opposite in language [9] but its negation is 'defined as its unique antonym', $p' = p^a$, then and since it is $p \cdot p^a = p \cdot p' < (p \cdot p')'$, provided the relation < is transitive (theorem of Non Contradiction [2]), nothing can be said on the self-contradictory, impossible, conjunction $p \cdot p'$. Hence, 'coincidence as conjunction' should be just considered for those premises

$p$ with opposites $p^a$ different from its also existing negation $p'$. In any case, nothing can follow from the impossible statement $p \cdot p'$, and only the conjunction $p \cdot p^a$ can be considered. It is insensate to consider *coincidentia* as the linguistic conjunction between $p$ and its negation $p'$.

3.2. From $p \cdot p^a < p$ [2], it follows $p' < (p \cdot p^a)'$. Hence, $(p \cdot p^a)'$ always has the inferential lower bound $p'$, and also the lower bound $(p^a)'$, since it is also $p \cdot p^a < p^a$. Provided it were $p \cdot p^a < p'$, it will follow (under transitivity) $p \cdot p^a < (p \cdot p^a)'$, that is, the conjunction $p \cdot p^a$ will be self-contradictory, impossible. Thus, it should be considered that it is but $p \cdot p^a </ p'$, that $p$ is a conjecture from the 'coincidence' $p \cdot p^a$. Notice that in Ortho-lattices and, in particular, in Boolean algebras, $p \cdot p^a < p' \Rightarrow p \cdot (p \cdot p^a) = (p \cdot p) \cdot p^a = p \cdot p^a < p \cdot p' = 0$, that is, $p \cdot p^a = 0$; in all Ortho-lattice $p$ and $p^a$ are incompatible, they have nothing in common.

On which conditions such conjunction is also a conjecture from p? Were it $p < (p \cdot p^a)'$, since it follows $(p \cdot p^a)'' < p'$, in the case the negation is weak or strong at $p \cdot p^a$, that is, $p \cdot p^a < (p \cdot p^a)''$, and with local transitivity in the triplet $((p \cdot p^a), (p \cdot p^a)'', p')$, it will follow $p \cdot p^a < p'$; thus, it should be $p </ (p \cdot p^a)'$.

That is, with suitable local transitivity and weak or strong negation, and as it was shown in paragraph 2.2, also $p \cdot p^a$ is a conjecture from p, and, *mutatis mutandis* is also a conjecture from $p^a$. Anyway, it can be stated that understanding the coincidence of opposites as its linguistic conjunction, it is not a 'principle'.

Thus, if the conjunction of opposites is always a hypothesis for the premise, on some (actually weak) conditions is a conjecture from it; this is the case, for instance, in Ortho-lattices and De Morgan algebras and, in particular, in Ortho-modular lattices and Boolean algebras . Obviously, since it is not $p \lozenge (p \cdot p^a)$ but $p \cdot p^a < p$, the conjunction of opposites is not a speculation.

It should be noticed that what has been said on the Unity of Opposites sheds some additional light upon the *Dialectic Synthesis*, previously analyzed in reference [5], with p the *thesis*, $p^a$ the *antithesis*, and $p \cdot p^a$ the *synthesis*. Actually, and in the current perspective from people's language and commonsense reasoning, it can't be surprising to jointly consider Dialectics, at the end coming from what is typical of conversation.

3.4. A note on the truth of the conjunction of opposites is still convenient. If t is a 'degree of true' [2,8], from $p \cdot p^a < p$ it follows $t (p \cdot p^a) = T (t (p), t (p^a)) \leq \min (t (p), t (p^a)) \leq t (p)$, and also $\leq t (p^a)$, with T an operation able to represent 'and'. That is, the degree of true of the conjunction of opposites is smaller than both degrees of p and $p^a$.

Since $p^a < p' \Rightarrow t (p^a) \leq t (p')$, such degree is also smaller than the degree up to which not-p is true. Notice that were p completely true, $t (p) = 1$, then is $t (p') = 0$, and also $t (p . p^a) = 0$; instead, if p is completely false, $t (p) = 0$, is $t (p') = 1$ and $t (p \cdot p^a)$ can have any value between 0 and 1. Provided it is $t (p) = 0.7$, for instance, since $t (p') = 1 – 0.7 = 0.3$, it will be $t (p \cdot p^a) \leq 0.3$, and if it is $t (p) = 0.3$ it is $t (p \cdot p^a) \leq 0.7$.

Hence, as smaller is the degree of true of p, greater can be that of the conjunction of opposites whose degree, notwithstanding, will remain unknown unless operation T is specified and the degree up to which $p^a$ is true is known; a degree on which it is only known $t (p^a) \leq 1 – t (p)$. Notice that $t (p) = 0.7 \Rightarrow t (p^a) \leq 0.3$, and $t (p) = 0.3 \Rightarrow t (p^a) \leq 0.7$, newly, as small is p true greater can be $p^a$ true.

Since what can be previously known is t (p), expressing $p^a$ through p can be interesting, as it was said in the Introduction; it is something requiring a jump from just considering the symbols p and $p^a$, up to translating such statements into membership functions depending on its linguistic expressions for all x in the universe of discourse X, that is, up to p (x) = 'x is p', and $p^a$ (x) = 'x is $p^a$'.

## 4. A LINGUISTIC AND VERY SIMPLE EXAMPLE

Let's consider what can be concluded from the linguistic statement p (x) = 'x is neither big nor small', for elements x in the unit interval [0, 1], and by means of the unity of opposites understood as in the former section. Such statement, shortened p (x) = 'x is medium', can be represented by a fuzzy proto-form {6}:

$$T\ (N\ (m_{big}\ (x)),\ N\ (m_{small}\ (x))) = m_{medium}\ (x),$$

where T represents the conjunction 'and', N the negation (supposed strong , as it is usually with fuzzy sets), and $m_{big}$, $m_{small}$ the respective membership functions to the fuzzy sets whose linguistic labels are 'big' and 'small', respectively.

Since membership functions are but measures of the meaning of the linguistic labels [2,6], the former equation represents how the meanings of 'not big' and 'not small' interlock in X to represent 'medium'. It is from such interlock that a common ground between p and $p^a$ can follow.

Supposed $m_{big}\ (x) = x$, it should be $m_{small}\ (x) = m_{big}\ (s\ (x))$, with a symmetry s in [0, 1] [9]; this formula shows how $p^a$ can be expressed through p [2,6,9].

Because of the coherence condition between the negation and the antonym, functions N and s should verify:

$$m_{small} \leq N \circ m_{big} \Leftrightarrow m_{big}\ (s\ (x)) \leq N\ (m_{big}\ (x)) \Leftrightarrow s\ (x) \leq N\ (x), \text{ for all } x \text{ in } [0, 1].$$

Taking, for simplicity, s (x) = N (x) = 1 – x, and T = product, the former proto-form is fully specified by

$$m_{medium}\ (x) = (1 - x)\ .\ (1 - (1 - x)) = x\ .\ (1 - x),$$

a membership function representing those elements x in [0, 1] that are neither big nor small; that is, the fuzzy set labeled 'medium'.

Accordingly with what is in section 3.2, and provided it were p the reasoning's premise, it can be concluded that 'x is medium' is a conjecture from both statements 'x is not big', and 'x is not small'; concretely, 'x is medium' is a hypothesis for such statements. Notice that now the negation is strong, and the inference relation $<$, particularized in the linear order $\leq$ of the unit interval, is transitive.

## 5. ON THE ACTUAL EXISTENCE OF THE *CONTRADICTIO OPOSSITORUM*

What is just shown, by considering the Unity of Opposites of p and $p^a$ as its linguistic conjunction is that, provided $p \cdot p^a$ is not self-contradictory, it is a hypothesis for the premise; a hypotheses that if negation is weak or strong and the inference relation < is transitive for, at least, the involved elements, is a conjecture of the premise p. In this way, the conjunction of opposites facilitates a hypothesis as synthesis, allows to conjecturing an explanation of p whose suitability for the problem on consideration is, in principle, something to be studied. Were $p \cdot p^a$ not suitable, it still remains open a new search for other hypotheses h for p, but now pivoting on $p \cdot p^a$ instead than in p. If it were $h < p \cdot p^a$, since it is always $p \cdot p^a < p$, under transitivity, it will follow $h < p$; were $p \cdot p^a < h$, or $p \cdot p^a \lozenge h$, then it will be necessary to directly prove if it is $h < p$, or it is not.

In any case, such interpretation of the conjunction of opposites, provided it is not self-contradictory, can be useful for conjecturing a first hypothesis explaining p.

5.1. Once the coincidence of opposites is seen through 'linguistic conjunction', it is unavoidable to know where, in the corresponding universe of discourse, it actually exists. That is, a ground where the unity of opposites actually holds should be specified in the corresponding universe; a ground consisting in the set of those elements $x \in X$ for which it actually holds '$p(x)$ and $p^a(x)$'. It should be recalled that in Boolean algebras is $p < p' \Leftrightarrow p = 0$; thus, were $p \cdot p^a$ self-contradictory, the former subset of X is empty. Hence, the ground's non emptiness always requires to be checked; were it empty, the coincidence is fictitious, it is for nothing.

Consequently, a non-empty ground for the 'coincidence' should be previously defined for actually supporting the non self-contradiction of the conjunctive statement $p \cdot p^a$. Without such ground, 'coincidence' is just something 'out of things'; a not realistic, but metaphysical concept. Let's consider some possibilities towards defining such ground.

5.2. In those frequent situations in which what is in reasoning comes from perception, there is a way for founding a ground through 'indistinguishability'; that is, by means of a relation I of T-Indistinguishability [2], or fuzzy equivalence, in the form I ($m_P$ (x), $m_P{}^a$ (x)) = Degree up to which p (x) is indistinguishable from $p^a$ (x), with p (x) = 'x is P', and $p^a$ (x) = 'x is $P^a$', for x in the universe of discourse X, and $P^a$ an antonym of P. That is, for assuring the biggest than possible perceptive 'distinguishability' between the opposites p and $p^a$; a distinguishability that can be considered by means of D (a, b) = 1 – I (a, b), a function that often is a distance [10]. As bigger is D (a, b) $\in$ [0, 1], as smaller is I (a, b) $\in$ [0, 1]; as more distinguishable are x and y, less indistinguishable are between them.

Thus, the degree I ($m_P$ (x), $m_P{}^a$ (x)) $\in$ [0, 1] should be as small as possible; something that is to be checked before conjunction. The set of those x in X such that the indistinguishability between p (x) and $p^a$ (x) can be lower than some given and small threshold $\varepsilon > 0$, will constitute the $\varepsilon$-support on which the conjunction of opposites is based. Such $\varepsilon$-support is the set {x $\in$X; I (p (x), $p^a$ (x)) ≤ $\varepsilon$}.

In this way, for instance, and in the former example of section 3, with I (a, b) = 1 - |a - b|, it will be I (a, b) =1 - |a – (1 – b)| = 1 - |a + b - 1|. Then, for each small threshold $\varepsilon \leq 1/2$ of indistinguishability that can be fixed, inequality I (a, b) ≤ $\varepsilon$ ⇔ |a + b - 1| ≤ 1 - $\varepsilon$, changing number a by $m_P$ (x) and number b by $m_P{}^a$ (x), will represent the set of those points x for which 'big' is indistinguishable from 'small' at the threshold $\varepsilon$. Thus, it is this subset, where it is D (a, b) ≥ 1 - $\varepsilon$, that supporting the conjunction of opposites; is in such subset were it has sense.

It should be remembered that with p' (x) = 'x is P' (not P)', is obtained a type of *Fuzziness* meaning the indistinguishability between p and not-p [7]. If $p^a$ = p', the indistinguishability between opposites is but a numerical index for the 'fuzziness' of p; thus, it seems that considering the indistinguishability between p (x) and $p^a$ (x) has some linguistic sense.

What all that can additionally imply from the reasoning's point of view? By fixing one of the arguments in I, it is obtained a membership function $I_b$ (a) = I (a, b) of the fuzzy set with the linguistic label 'indistinguishable from b', a curve in the surface given by I. Notice that since I is symmetric, I (a, b) = I (b, a) for all a and b in [0, 1], it is $I_b$ (a) = $I_a$ (b). In the former example, for instance, it is $I_{0.7}$ (a) = 1 - |0.7 - a| = 0.3 + a, if 0 ≤ a ≤ 0.7, or 1.7 - a, if 0.7 < a ≤ 1.

Since by its T-transitive law [2], I verifies: T (I (a, b), I (b, c)) ≤ I (a, c), for all triplet a, b, c in [0, 1], it is T ($I_a$ (b)), I (b, c)) ≤ $I_a$ (c), an inequality that can be seen as a 'principle' of approximate inference interpreting that $I_a$ (b) is the strength in which b is indistinguishable from a: The operation with T between this strength and the degree up to which b is indistinguishable from c, is a lower bound for the strength in which c is indistinguishable from a. It can be interpreted, in the former example, as: The strength some b has with a as 'not big', operated under a 'conjunction' T, with how much is b indistinguishable with another element c, is an inferior bound of the strength as 'not big' between elements b and c. In triplets (a, b, c), the 'distinguishability' between a and c, depends on those between a and b, and between b and c; distinguishability between a and c, appears related with what surrounds them (b).

Notice that if the first strength is 1 (a is totally indistinguishable from b), then it is I (b, c) = $I_c$ (b) ≤ I (a, c) = $I_c$ (a), the first strength is smaller than the second. Were it such strength 0 (not indistinguishable at all, or totally distinguishable), it will result the not at all informative conclusion 0 ≤ $I_c$ (a).

What is shown in this paragraph 5.2 opens a possible new way towards looking at the ground as a fuzzy set [2,8], something formerly presented [6] in relation with the Dialectic Synthesis, and that was promptly applied to the Human Trafficking problem [12]. It is an application seeming to reinforce the validity of what is presented in reference [6], and also in this paper.

5.3. There are more different, simpler, not depending on any threshold, and perhaps clearer forms for specifying a ground supporting the conjunction of opposites. Let's consider the three following forms.

The first, is by considering the complement of the set {x ∈X; $m_P$ (x) = $m_P{}^a$ (x)}, that is, the set of those x in which P and $P^a$ are different. For instance, in the former example is the set {a ∈[0, 1]; a = 1 – a}$^c$ = {1/2}$^c$ = [0, ½) U (1/2, 1]. Notice that point ½ is but the Max Black's 'separation' [8] between the linguistic terms 'not small' and 'not big'; only in the points different from it has sense to consider the conjunction 'not small and not big' = 'medium'. Notwithstanding, this form does not depend on the perceptive possibility of distinguishing p (x) and $p^a$ (x) as it is in the form shown in 5.2, and it is even free of any previously chosen threshold. On what both forms depend is in the careful design of the membership functions $m_P$ and $m_P{}^a$, as well as on that of symmetry s and negation N.

The second form appears when, for some reason, considering the grounds in which points are either less P than $P^a$, or less $P^a$ than P can be interesting. Then, the sets to be respectively considered are $\{x \in X;\ m_P(x) \leq m_P{}^a(x)\}$, or $\{x \in X;\ m_P{}^a(x\neq) \leq m_P(x)\}$, whose intersection is the first set. In the former example, this respective sets are the intervals [0, ½) and (1/2, 1]. It is threshold-free.

The third form, also threshold-free, comes from considering the set $\{x \in X;\ T(m_P(x), m_P{}^a(x)) \neq 0\}$; notice that were 'crisp' the meanings of P and $P^a$, that is, its corresponding membership functions only take the values 0 or 1 [2,6], such subset is but the intersection $m_P{}^{-1}(1) \cap m_P{}^a(1)$ of the subsets respectively specified in X by P and $P^a$. In the former example and with s = N, such subset is $\{x \in [0, 1];\ T(x, 1 - x) = 0\}^c$, specified once T is chosen; were T = product, $x \cdot (1-x) = 0 \Leftrightarrow x = 0$ or $x = 1$, with which the subset is $\{x \in [0, 1];\ x \neq 0, x \neq 1\} = (0, 1)$. Were $s(x) = \sqrt{(1 - x^2)}$, it is also $x \cdot \sqrt{(1 - x^2)} = 0 \Leftrightarrow x = 0$ or $x = 1$.

5.4. Thus, there is no a single form to specify a ground. For this reason, it should be previously decided on which form the ground where the conjunction of opposites can be effectively considered, will be actually specified. The Unity of Opposites needs a real support without which the reasoning can be, as it was said, out of things, metaphysical.

5.5. Concerning not regular linguistic terms q, those for which no antonym is known in language, it is not strictly necessary to define $q^a$ = q' since with symmetries $s: X \rightarrow X$ it is possible to obtain antonyms for them. For example, in X = [0, 1] and in mathematical language, there is not an antonym of the statement q = 'less than 0.3' that, in principle, is not necessary. Nevertheless, its negation is q' = 'more than 0.3', and antonyms of q can be specified by means of $m_q{}^a(x) = m_q(s(x))$, with suitable symmetries s of [0, 1]. For instance, with the symmetry

$s(x) = (1 - x^2)^{1/2}$, it is $m_q((1 - x^2)^{1/2}) = m_{[0, 0.3]}((1 - x^2)^{1/2})$, equal to 1 if $0 \leq (1 - x^2)^{1/2} \leq 0.3 \Leftrightarrow \sqrt{0.91} \leq x \leq 1$, and equal to 0 if $0.3 < (1 - x^2)^{1/2} \leq 1 \Leftrightarrow 0 \leq x < \sqrt{0.91}$.

Thus, since $q^a$ is specified by the interval $[\sqrt{0.91}, 1]$, linguistically $q^a$ is the statement 'more than √0.91', different from the negation 'more than 0.3', and since it is $m_{[\sqrt{0.91}, 1]} \leq m_{[0.3, 1]} \Leftrightarrow m_q{}^a \leq m_{q'}$, $q^a$ is coherent with q'; hence, $q^a$ = 'more than √0.91' is a good possible antonym of q = 'less than 0.3' that, in this way, results to be a regular statement.

Hence, the not regularity of a statement can be seen as provisory; it is so up to when language requires more expressivity, and consequently, creates and gives a name to some opposite for it. Nevertheless, in the former example such language's requirement does not actually exist, since all can be well expressed with the words 'less', 'more', and the mathematical concept exhibited by the symbol ≤. It is a setting in which 'mathematical language' is sufficiently expressive.

Let's finally remark that with the symmetry $s(x) = 1 - x$, and *mutatis mutandis,* is just obtained $q^a = q'$.

## 6. CONCLUSIONS

6.1. The current paper, that is not a purely Logic's paper, just tries to translate into a symbolic skeleton of reasoning, endowed with a short number of definitions, laws and properties (minimally required by commonsense reasoning expressed in a natural language), a form on which the Unity of Opposites can be understood. It could, perhaps, be said that within a naïve and initial 'formalization', an approach to *coincidentia opositorum* is introduced with just a minimal, formal skeleton of reasoning showing a wide validity in commonsense reasoning. Such skeleton can be seen as a theoretic 'working framework'.

Why presuming few laws? It is because of not all the laws usually presumed in Algebraic Logic can't be universally managed in commonsense reasoning. Those are, for instance, the commutative law of conjunction, the associative of disjunction, the interpretation of the conditional statement $p < q$ as 'not p or q', etc [2.4.8], etc. Analogously, some hypothetical axioms presumed in some 'logical' approaches to Dialectics [13, 14] are not always present in ordinary language and commonsense reasoning; this is a main reason for adopting some theoretic working-framework, offered by a 'skeleton' based on a short number of laws. A skeleton of which the one presented here is but an instance.

Nevertheless, the minimal number of laws presumed for the presented skeleton of reasoning, allow to prove, for instance, that deducing and refuting are monotonic, abducing and conjecturing are anti-monotonic, and speculating is just non-monotonic [2,8,16], as well as that with transitivity both consequences and not self-contradictory hypothesis are conjectures, that weak speculations can be deductively attained and that in finite Boolean algebras strong speculations can be attained by a computer program [4] and through a zigzag of mixed chains of inference.

It can be also proven that the old *principles* of Non-Contradiction and Excluded Middle are but *theorems*, by just interpreting the Aristotelian term 'impossible' as self-contradictory; the first was already used in 3.1. For instance, that of Non-Contradiction, $p \cdot p' \leq (p \cdot p')'$, is proven by:

$$p \cdot p' < p \;\;\&\;\; p \cdot p' < p' \Rightarrow p' < (p \cdot p')' \Rightarrow p \cdot p' < (p \cdot p')',$$

a very short proof contradicting the own words of Aristotle, who wrote that Non-Contradiction is a principle because of it 'can't be submitted to proof'. The theorem of Excluded Middle, a 'principle' not so clearly stated by Aristotle and questioned by some logical approaches as well as by Marxian Dialectics, is proven by:

$$p < p + p' \;\&\; p' < p + p' \Rightarrow (p + p')' < p' \;\&\; p' < p + p' \Rightarrow (p + p')' < p + p' \Rightarrow (p + p')' < ((p + p')')',$$

that is, not (p or not p) is impossible, or 'p or not p' is not impossible.

Both theorems [2,4,8,16] follow up from a short number of previous 'principles' concerning conjunction, disjunction, negation and the local transitivity of <. These principles are the seven before presumed laws,

1) $p < p$; 2) The Rule of Modus Ponens; 3) $p < q \Rightarrow q' < p'$; 4) $p \cdot q < p$, and $p \cdot q < q$; 5) $p < p + q$, and $q < p + q$; 6) negation (‘) is not wild at all q; 7) $<$ is locally transitive,

that, as defining the cited skeleton, could be called the *First Formal Principles of Commonsense Reasoning.*

6.2. The conjunction $p \cdot p^a$ reflects an intermediate, but not necessarily symmetrical position, between what is p and what is $p^a$ in the corresponding ground; it means considering what each x in the universe of discourse shows as being both p and $p^a$. Is a kind of counterbalance between opposites at each x; can be seen as an ‘integration, or synthesis, of extremes’ and, actually, it is not too far from what is behind the conjectural methodology followed by those philosophers known, *à la* Ferrater Mora [17], as ‘integrationists’.

In sum, the Unity of opposites seems to be a ‘root of thought’ that, mainly from Nicholas of Cusa to Friedrich Hegel, arrived up to support Karl Marx and Friedrich Engels Dialectical Materialism, as well as José Ferrater Mora’s Philosophical Integrationism [11], although not in the same form in all cases that, anyway, are but different philosophical approaches towards conjecturing explanations.

6.3. Since $p \cdot p^a$ is but a hypothesis for p, an ending comment on a different way, or method, for attaining conjunctive hypotheses from p is also suitable. Obviously, for all statement q is $p \cdot q < p$; thus provided $p \cdot q$ is not self-contradictory, it is a hypothesis for p.

Notwithstanding, when thinking on a question concerning p, and if for it p should be explained, it seems reasonable to consider statements q related to p in the question’s context. If a possibility is tacking $q = p^a$, that is refuted by p, there is also that of considering q as a weak or strong speculation s (p) from p, and directly coming through reflecting on a question on p posed in some particular context. It is a way that possibly all researcher can recognize in her/his usual praxis [2,8].

In any case, either $p^a$ or s (p) conducts to explain p; $p \cdot p^a$ and p · s (p) are but different alternatives for attaining first hypotheses for p since, as it was shown, $p \cdot p^a$ is not a speculation from p. Let's again recall that the Non-Contradiction's theorem shows the absurd of considering p · p' because of, even verifying $p \cdot p' < p$, it is self-contradictory.

At the end, as it was said at the end of paragraph 3.2 and even contradicting *Lenin*, the conjunction of opposites is not a 'principle' of reasoning but a method of explanation that, in addition, is not unique.

To actually finishing the paper, let's remember what is said at the end of section 1: Everything is here tried to be made as simple as possible, but not simpler.

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